Periodic solutions of a two-degree-of-freedom autonomous vibro-impact oscillator with sticking phases

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Abstract This contribution explores the free dynamics of a simple two-degree-of-freedom vibro-impact oscillator. One degree-of-freedom is limited by the presence of a rigid obstacle and periodic solutions involving one sticking phase per period (1-SPP) are targeted. A method to obtain such orbits is proposed: it provides conditions on the existence of 1-SPP as well as closed form solutions. It is shown that 1-SPP might not exist for a given combination of masses and stiffnesses. The set of 1-SPP is at most a countable set of isolated periodic orbits. The construction of 1-SPP requires numerical developments that are illustrated on a few relevant examples. Comparison with nonlinear modes of vibration involving one impact per period (1-IPP) is also considered. Interestingly, an equivalence between 1-SPP and a special set of isolated 1-IPP is established. It is also demonstrated that the prestressed system features sticking phases of infinite duration.

Keywords vibration analysis; impact dynamics; sticking phase; periodic solutions; two degrees-of-freedom

1. Introduction

In the context of vibration and modal analysis of vibro-impact oscillators, nonlinear modes of vibration with non-grazing impact are explored in [7, 8, 10, 12, 14, 15]. In these works, one degree-of-freedom (dof) is unilaterally constrained by the presence of a rigid foundation: the dynamics is purely linear when the contact constraint is not active, and governed by an impact law otherwise. Accordingly, the contact force arising when the system interacts with the foundation is a periodic distribution of Dirac deltas. Instead, the present work pays attention to periodic solutions involving finite time sticking contact phases — thus discarding impulse-driven dynamics reported in the previous works — during which the contacting mass rests against the obstacle. Is considered a simple mechanical system of two masses

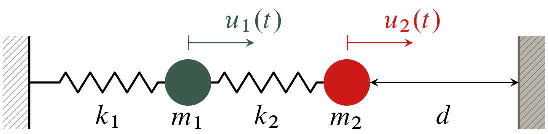

Figure 1: Investigated two-degree-of-freedom vibro-impact system with $d > 0$

linearly connected through two springs, as depicted in Figure 1, and the dynamics reads

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{r} \tag{1.1a}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0 \tag{1.1b}$$

$$u_2(t) \leq d, \quad R(t) \leq 0, \quad (u_2(t) - d)R(t) = 0, \quad \forall t \tag{1.1c}$$

$$\dot{\mathbf{u}}^\top \mathbf{M}\dot{\mathbf{u}} + \mathbf{u}^\top \mathbf{K}\mathbf{u} = \mathbf{E}(t) = \mathbf{E}(0) \tag{1.1d}$$

where

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad ; \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \quad ; \quad \mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \quad ; \quad \mathbf{r}(t) = \begin{pmatrix} 0 \\ R(t) \end{pmatrix}.$$

The two restoring forces stem from the action of stiffnesses k_1 and k_2; the corresponding masses are m_1 and m_2. Quantities u_j, $\dot{u}_j$, and $\ddot{u}_j$ represent the displacement, velocity, and acceleration of mass j, $j = 1, 2$, respectively. The gap d separates the obstacle and the equilibrium position of the second mass[1]. $R(t)$ is the reaction force of the wall on mass 2. Matrices $\mathbf{M}$ and $\mathbf{K}$ are symmetric positive definite so that there is a matrix $\mathbf{P}$ of eigenmodes which diagonalizes them simultaneously, that is $\mathbf{P}^\top\mathbf{M}\mathbf{P} = \mathbf{I}$ and $\mathbf{P}^\top\mathbf{K}\mathbf{P} = \mathbf{\Omega}^2 = \mathbf{diag}(\omega_i^2)|_{i=1,2}$ where $\mathbf{I}$ is the identity matrix in $\mathbb{R}^2$ and ω_i^2, $i = 1, 2$ are the eigenfrequencies of the linear system without unilateral contact. For the well-posedness of the initial-value problem with conserved energy, see [3, 11]. When external forcing is introduced, sticking phases are known to emerge as limits of a chattering sequence [2, 5, 6, 9]. There is no source term in this work but sticking periodic solutions can still occur.

The natural way to obtain periodic solutions to System (1.1) is to look for the fixed points of the associated first return map (FRM) which can be exhibited almost explicitly. Closed forms are obtained as in [8] except for an unknown parameter: the free flight time which is a root of an explicit function h. The roots of h belong to a countably infinite set of initial data which yield one-sticking-phase-per-period periodic solutions (1-SPP) if and only if the constraint $u_2 < d$ is satisfied during the free flight. This extends to the prestressed case $d \leq 0$.

The paper is organized as follows. In Section 2, the "sticking phase" definition is provided and the conditions on its occurrence are stated. Then, necessary conditions satisfied by 1-SPP are given on the period through the free flight duration s which is the key parameter, root of h. Furthermore, when s is known, the corresponding 1-SPP is expressed in a closed form. The method and numerical examples are described in Section 3 in order to find all 1-SPP. Mathematical proofs and comments are detailed in Sections 4, 5 and 6. More precisely, Section 4 deals with the structure of the solution space with a sticking phase. Section 5 is devoted to prove Theorem 2.1 on 1-SPP. The existence of an infinite set of admissible initial data satisfying the constraint $u_2 \leq d$ near the sticking phase is proven in Section 6. The existence of 1-SPP satisfying the constraint $u_2 \leq d$ during the whole period remains an open problem. The prestressed case with $d \leq 0$ is discussed in Section 7. Section 8 concludes the paper.

2. Main results

The sticking phase is first defined with necessary and sufficient conditions on its occurrence in Section 2.1. Then, in Section 2.2, periodic solutions with one sticking phase per period, the so-called 1-SPP, are characterized with necessary conditions needed to list them all.

2.1. Sticking phase

A sticking phase occurs when the second mass stays at $u_2 = d$ during a finite time interval.

Definition 2.1 [Sticking phase and its duration] *Let* $\mathbf{u}$ *be the solution to System* (1.1). *A* sticking phase *arises if there exist* $t_0 \in \mathbb{R}$ *and* $\mathcal{T} > 0$ *such that*

$$u_2(t) = d, \quad \forall t \in [t_0\,; t_0 + \mathcal{T}]. \tag{2.1}$$

Moreover, when there exists $0 < \delta \ll 1$ *such that* $\forall t \in]0\,; \delta[$

$$u_2(t_0 - t) < d \quad \text{and} \quad u_2(t_0 + \mathcal{T} + t) < d, \tag{2.2}$$

then t_0 *is the starting time and* $\mathcal{T}$ *is the duration of the sticking phase.*

The central reference [3] is used throughout the paper: existence is recalled, uniqueness and continuous dependence to the initial data is proved. Moreover, in the conservative case, it is shown that there is no

[1] d is the algebraic *distance* between the equilibrium position of mass 2 and the rigid wall, and might be negative with prestress.

impact accumulation. Accordingly, before and after the sticking phase, Condition (2.2) is sufficient to define the beginning and the end of a sticking phase of finite duration. More precisely, it is proved in [3], Proposition 19 and Subsection 6.4, that the impact times are isolated for the perfectly elastic impact law.

In the present work, the finite duration of the sticking phase for $d > 0$ is a consequence of Theorem 2.1. This is not always true, as for example with the prestressed case $d \leq 0$ detailed in Section 7. Such conditions are well known to be related with the sign of the acceleration [6] and are specified and developed for our 2-dof system as in [7].

Theorem 2.1 [Sticking contact] *There exists a sticking phase exactly starting at time t_0 and persisting on its right neighbourhood if and only if:*

1. $u_2(t_0) = d$, $\dot{u}_2^-(t_0) = 0$, $u_1(t_0) > d$, *or*
2. $u_2(t_0) = d$, $\dot{u}_2^-(t_0) = 0$, $u_1(t_0) = d$, $\dot{u}_1(t_0) > 0$.

The second case where $u_1(t_0) = d$ and $\dot{u}_1(t_0) > 0$ corresponds to the beginning of the sticking phase. The duration of the sticking phase $\mathcal{T}$ then only depends on

$$v = \dot{u}_1(t_0) \tag{2.3}$$

with

$$\mathcal{T}(v) = \frac{2}{\omega}\arctan(\xi v) \quad \textit{where} \quad \omega = \sqrt{\frac{k_1 + k_2}{m_1}} \quad \textit{and} \quad \xi = \frac{\sqrt{(k_1 + k_2)m_1}}{k_1\, d}. \tag{2.4}$$

The state of system at the end of the sticking phase is

$$u_2(t_0 + \mathcal{T}) = d, \quad \dot{u}_2^-(t_0 + \mathcal{T}) = 0, \quad u_1(t_0 + \mathcal{T}) = d, \quad \dot{u}_1(t_0 + \mathcal{T}) = -v. \tag{2.5}$$

Moreover, the regularity of the curve $\{(u_2(t), \dot{u}_2(t)), t \in \mathbb{R}\}$ is $C^{1.5}$ at the point $(u_2, \dot{u}_2) = (d, 0)$ which corresponds to the time interval $[t_0, t_0 + \mathcal{T}]$.

The loss of regularity at the sticking point can be seen explicitly in Figure 2. It is the least smooth point

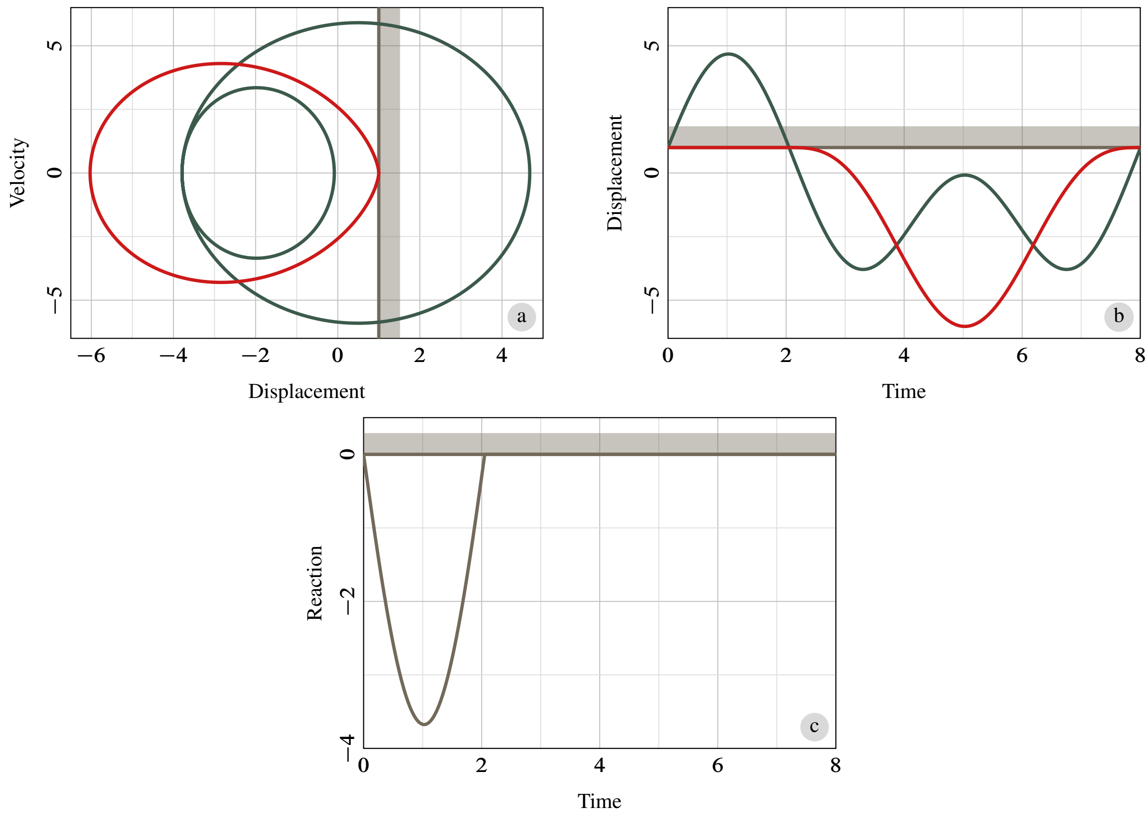

Figure 2: Admissible 1-SPP with a singularity 1.5 at the intersection between red line and wall. (a) Orbits $\{(u_i(t), \dot{u}_i(t)),\ t \in \mathbb{R}\}$, $i = 1, 2$. (b) Displacements $t \mapsto u_i(t)$, $i = 1, 2$. (c) Reaction of the wall $t \mapsto R(t)$

in the curve $\{(u_2(t), \dot{u}_2(t)),\ t \in \mathbb{R}\}$. Locally, the orbit is very similar to the graph of $t \mapsto (d, |t|^{1.5})$.

Elsewhere, the curve is analytic.

Incidentally, the proof of Theorem 2.1 leads to an explicit classification of all possible contact patterns. Since there is no accumulation of impacts, the latter are isolated and only three distinct configurations arise. Assume a contact at $t = t_0$, then $u_2(t_0) = d$ and the contact is

1. an impact if $\dot{u}_2^-(t_0) > 0$,
2. a grazing contact if $\dot{u}_2(t_0) = 0$ and $R(t) = 0$ for all $t \approx t_0$, or
3. a sticking contact if $\dot{u}_2(t_0) = 0$ and $R(t) < 0$ for some $t \approx t_0$.

The velocity of mass 2 shall be discontinuous and $\dot{u}_2^\pm(t_0)$ denotes its left/right limit. Energy conservation implies $\dot{u}_2^+(t_0) = -\dot{u}_2^-(t_0)$; in particular, when the incoming velocity vanishes, that is $\dot{u}_2^-(t_0) = \dot{u}_2^+(t_0) = \dot{u}_2(t_0) = 0$, the velocity is continuous.

Sticking contact and grazing contact are the two critical contact events [6]. In general, it is challenging to know whether a zero pre-velocity impact generates a reaction of the wall without further information. For the 2-dof system, simple criteria are given on the position and velocity of mass 1 to distinguish grazing from sticking:

1. An impact yields an instantaneous bounce with $\dot{u}_2^+(t_0) = -\dot{u}_2^-(t_0) < 0$.
2. A grazing contact means that the trajectory would not change without the wall. As a corollary of Theorem 2.1, the data at time t_0 are either $u_1(t_0) < d$, $u_2(t_0) = d$, and $\dot{u}_2(t_0) = 0$ or $u_1(t_0) = d$, $\dot{u}_1(t_0) \leq 0$, $u_2(t_0) = d$, and $\dot{u}_2(t_0) = 0$. Moreover, there exists $\varepsilon > 0$ such that $R(t) = 0$ for all $t \in \,]t_0 - \varepsilon\,; t_0 + \varepsilon[$.
3. A sticking contact phase can be divided into three events:
   (a) its beginning: $u_1(t_0) = d$, $\dot{u}_1(t_0) > 0$, $u_2(t_0) = d$, and $\dot{u}_2(t_0) = 0$. There exists $\varepsilon > 0$ such that $R(t) = 0$ for all $t \in \,]t_0 - \varepsilon\,; t_0[$ and $R(t) < 0$ for all $t \in \,]t_0\,; t_0 + \varepsilon[$;
   (b) its resting phase: $u_1(t_0) > d$, $u_2(t_0) = d$, and $\dot{u}_2(t_0) = 0$. There exists $\varepsilon > 0$ such that $R(t) < 0$ for all $t \in \,]t_0 - \varepsilon\,; t_0 + \varepsilon[$;
   (c) its ending: $u_1(t_0) = d$, $\dot{u}_1(t_0) < 0$, $u_2(t_0) = d$, and $\dot{u}_2(t_0) = 0$. There exists $\varepsilon > 0$ such that $R(t) < 0$ for all $t \in \,]t_0 - \varepsilon\,; t_0[$ and $R(t) = 0$ for all $t \in \,]t_0\,; t_0 + \varepsilon[$.

2.2. Periodic solutions with one sticking phase per period (1-SPP) The main results of this paper are concerned with the possible existence and computation of 1-SPP.

Definition 2.2 [One sticking phase per period solution] *A periodic function* $\mathbf{u}(t)$ *is called a 1-SPP, a periodic solution to* (1.1) *with one sticking phase per period, if there exists* $0 < \mathcal{T} < T$ *such that (up to a time translation)*

1. $u_2 = d$ *on* $[0\,; \mathcal{T}]$,
2. $u_2 < d$ *on* $]\mathcal{T}\,; T[$, *and*
3. $\mathbf{u}(T) = \mathbf{u}(0)$ *and* $\dot{\mathbf{u}}^-(T) = \dot{\mathbf{u}}^-(0)$.

Condition 2 above can be relaxed to $u_2(t) \leq d$ on $]\mathcal{T}\,; T[$ only. This yields admissible periodic solutions with potentially many grazing contacts and sticking phases.

In order to find and characterize all 1-SPP, the following notations are needed:

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \qquad \mathbf{P}^{-1} = \begin{bmatrix} P_{11}^{-1} & P_{12}^{-1} \\ P_{21}^{-1} & P_{22}^{-1} \end{bmatrix}, \tag{2.6}$$

$$\Phi_j(s) = \frac{\sin(\omega_j s)}{\omega_j(1 - \cos(\omega_j s))} = \frac{1}{\omega_j} \cot\Big(\frac{\omega_j s}{2}\Big), \tag{2.7}$$

$$a_{kj} = -P_{kj} P_{j1}^{-1}, \quad b_{kj} = \frac{a_{kj}}{\omega_j}, \quad \alpha_j = b_{1j} - b_{2j}, \quad \beta_j = b_{1j}, \tag{2.8}$$

$$w_k(s) = \sum_{j=1}^{2} a_{kj} \Phi_j(s) = \sum_{j=1}^{2} b_{kj} \cot\Big(\frac{\omega_j s}{2}\Big), \tag{2.9}$$

with $j = 1, 2$ and $k = 1, 2$. The interaction coefficients a_{kj} in [8] and in this work have opposite sign. Also, if a 1-SPP exists, then there is only one control parameter, the duration s of the free flight, which uniquely characterizes the 1-SPP through Theorem 2.2. The initial data and the period T are functions of s. Conversely, such initial data may correspond to ghost solutions [12] if u_2 exceeds d during free flight.

Theorem 2.2 [1-SPP characterizations] *Assume* $\mathbf{u}(t)$ *is a 1-SPP of System* (1.1)*, then:*

1. *The duration of the free flight* $s > 0$ *is necessarily a root of*

$$h(s) = w_1(s) - w_2(s) = \sum_{j=1}^{2} \alpha_j \cot\left(\frac{\omega_j s}{2}\right) = 0. \tag{2.10}$$

2. *The solution* $\mathbf{u}$ *corresponds to the initial data*

$$[u_1(0), u_2(0), \dot{u}_1(0), \dot{u}_2(0)] = [d, d, v, 0] \quad \text{where} \quad v = v(s) = d/w_1(s). \tag{2.11}$$

3. *The period* T *of* $\mathbf{u}$ *is a function of* s*:* $T(s) = s + \mathcal{T}(v(s))$ *where* $\mathcal{T}$ *is defined in* (2.4).
4. *The orbit is symmetric:* $\mathbf{u}(\theta + t) = \mathbf{u}(\theta - t)$, $\forall t$*, where* $\theta := \mathcal{T} + s/2$.

Remark 2.1. The parameter characterizing a 1-SPP could be the velocity v of the first mass at the beginning of the sticking phase. Fixing v fixes all the initial data (2.11) at the beginning of the sticking phase for a 1-SPP and then all 1-SPP. Accordingly, there is a one-to-one correspondence between the set of 1-SPP and the set of initial velocities $v = \dot{u}_1(t_0)$ yielding 1-SPP. However, 1-SPP closed-form expressions are simpler with s, parameter chosen in the remainder.

Remark 2.2. The set of 1-SPP is at most countable and corresponds to a subset of the roots of the analytic function $h(\cdot)$.

The roots of the quasi-periodic function $h(\cdot)$ are the first ingredients to be investigated in order to seek 1-SPP. In addition, the velocity of the first mass at the beginning of the sticking phase has to be positive, see Theorem 2.1. It is governed by the sign of $w_1(s)$.

The sticking phase is now exactly computed. Without loss of generality, assume $t_0 = 0$. The end of the sticking phase is the beginning of the free flight. Denoting $\underline{\mathbf{u}}$, the solution of the free flight with $\mathcal{T} = \mathcal{T}(s)$ leads to

$$(\underline{\mathbf{u}}, \dot{\underline{\mathbf{u}}})(\mathcal{T}) := (\mathbf{u}, \dot{\mathbf{u}})(\mathcal{T}), \tag{2.12}$$

$$\mathbf{M}\ddot{\underline{\mathbf{u}}}(t) + \mathbf{K}\underline{\mathbf{u}}(t) = 0, \quad \forall t \in]\mathcal{T}; T[. \tag{2.13}$$

A solution to Equations (2.12)-(2.13) is a physically admissible solution to System (1.1) if it satisfies the constraint

$$\underline{u}_2(t) < d, \quad \mathcal{T} < t < T. \tag{2.14}$$

If (2.14) is violated, then the 1-SPP is not admissible: this is a "ghost" solution [12]. Hence, introducing the following sets:

$$Z = \{s > 0,\ h(s) = 0\}, \tag{2.15}$$

$$Z^- = \{s \in Z \text{ and } w_1(s) < 0\}, \tag{2.16}$$

$$Z^0 = \{s \in Z \text{ and } w_1(s) = 0\}, \tag{2.17}$$

$$Z^+ = \{s \in Z \text{ and } w_1(s) > 0\}, \tag{2.18}$$

$$Z^{\text{ad}} = \{s \in Z^+ \text{ such that (2.14) is satisfied}\} \tag{2.19}$$

and $Z = Z^+ \cup Z^0 \cup Z^-$, the admissible free flight times s belongs to Z^+ which also corresponds to the "admissible" initial data. Furthermore, from the admissible initial data, the set of admissible 1-SPP has a one-to-one correspondence with Z^{ad}. Is the set Z^{ad} empty or not? This is a challenging question due to the global constraint (2.14). However, we can quantify the size of Z^+ which leads to solutions satisfying (2.14), at least near the sticking phase. The following assumption is needed to avoid that $Z^+ = \varnothing$, see Section 6.

Assumption 2.1 $\det\begin{pmatrix}\alpha_1 & \alpha_2 \\ \beta_1 & \beta_2\end{pmatrix} \neq 0.$

Notice that if Assumption 2.1 is violated and $(\alpha_1, \alpha_2) \neq 0$, $(\beta_1, \beta_2) \neq 0$ then $Z = Z^0$ and $Z^+ = \varnothing$ else Z^0 is a small set with at most two elements.

Theorem 2.3 [Countable infinity of Z^+] *If $\omega_1/\omega_2 \notin \mathbb{Q}$ then Z is countably infinite. Moreover, if Assumption 2.1 holds, Z^+ is also countably infinite.*

It is easy to show that Z is countably infinite when $\omega_1/\omega_2 \notin \mathbb{Q}$ since $h(\cdot)$ is quasi-periodic with many vertical asymptotes. The difficult part in Theorem 2.3 is to prove that Z^+ is also infinite. Incidentally, it turns out that Z^- is also infinite and more precisely that $\operatorname{card}(Z^+ \cap [0\,;A]) \sim \operatorname{card}(Z^- \cap [0\,;A])$ for large A. In other words, many roots of h do not correspond to 1-SPP. Not only 1-SPP are rare objects but among the roots of the function h, only a few correspond to admissible 1-SPP.

In the next Section, the procedure to find 1-SPP is detailed.

3. Examples

To construct 1-SPP, Theorems 2.1 and 2.2 are interpreted as follows: let $s > 0$ satisfy $h(s) = 0$ and $w_1(s) > 0$. Such s is a candidate to construct a 1-SPP $\mathbf{u}(t)$ of Problem (1.1) corresponding to the initial data $[u_1(0), u_2(0), \dot{u}_1(0), \dot{u}_2(0)]^\top = [d, d, +v, 0]^\top$ where $v = d/w_1(s)$, with a sticking phase on $[0\,;\tau]$ and then a free-flight on $[\tau\,;\tau+s]$ with $\tau = \mathcal{T}(s)$; more precisely:

- Sticking phase for $t \in [0\,;\tau]$: mass 2 sticks to the wall and mass 1 acts as a 1-dof linear oscillator.
- Free flight for $t \in]\tau\,;\tau+s[$: System (1.1a) is solved with "initial" data at time $\tau = \mathcal{T}(s)$ given as $[u_1(\tau), u_2(\tau), \dot{u}_1(\tau), \dot{u}_2(\tau)]^\top = [d, d, -v, 0]^\top$. The condition $u_2(t) < d$ is to be checked on the interval $]\tau\,;\tau+s[$ to obtain a real solution of Problem (1.1). Otherwise, an impact emerges before $\tau + s$ and the assumption of a free flight is violated on $]\tau\,;\tau+s[$ so that the corresponding $\mathbf{u}(t)$ is not a 1-SPP.

Accordingly, building a 1-SPP requires two numerical steps:

1. Compute the roots of $h(\cdot)$: Figure 3 depicts the set of roots as all the intersections of $h(\cdot)$ and the horizontal axis.
2. Check the admissibility of the associated solution, that is check if $v > 0$ and if $u_2(t) < d$ for all $t \in]\mathcal{T}(s)\,;\mathcal{T}(s)+s[$. From the symmetry of the solution during the free flight, it is sufficient to check $u_2(t) < d$ for all $t \in]\mathcal{T}\,;\mathcal{T}+s/2[$.

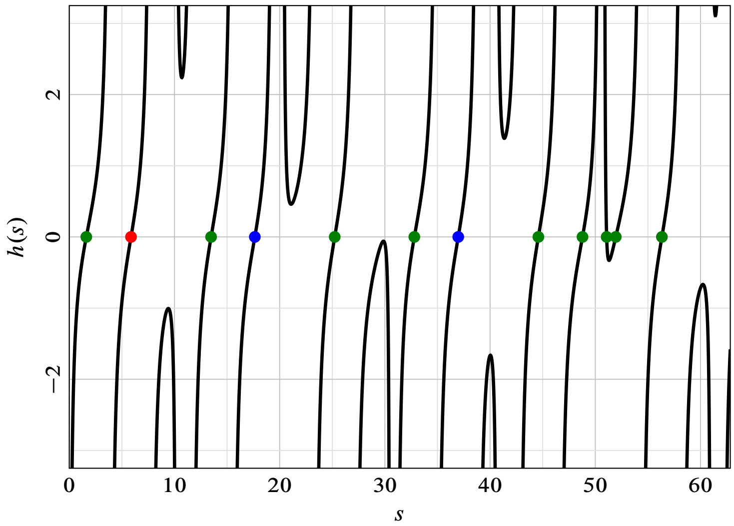

Figure 3: Set $Z = \{s \mid h(s) = 0\}$: red point: $s \in Z^{\text{ad}}$; blue points: $s \in Z^+$ but $s \notin Z^{\text{ad}}$; green points: $s \in Z^-$. The set of s corresponding to the admissible initial data are points in blue or red but only one point corresponds to a 1-SPP: the red point.

First numerical examples are provided with $m_1 = m_2 = 1\,\text{kg}$. The two periods of the unconstrained linear System (1.1a) are $T_1 \approx 10.17\,\text{s}$ and $T_2 \approx 3.88\,\text{s}$.

Figure 2 shows the simplest 1-SPP one can find: only one loop for the orbit of the second dof. This orbit is very smooth except at one point corresponding to the whole sticking phase. At this *sticking point*, a $C^{1.5}$-regularity only is achieved as discussed in Section 4.2. Various examples featuring other responses are introduced in Figures 4 and 5.

Many roots of h belonging to Z^+ do not correspond to 1-SPP. For instance, for $s \approx 17.97 \in Z^+$, the free-flight is not acceptable since the second mass penetrates the rigid obstacle, as pictured in Figure 6. The condition $s \in Z^+$ only stipulates that the non-penetration constraint (2.14) is satisfied near the sticking phase. Although Z^+ is infinite, it is challenging to find the set $Z^{\text{ad}} \subset Z^+$ yielding 1-SPP. For

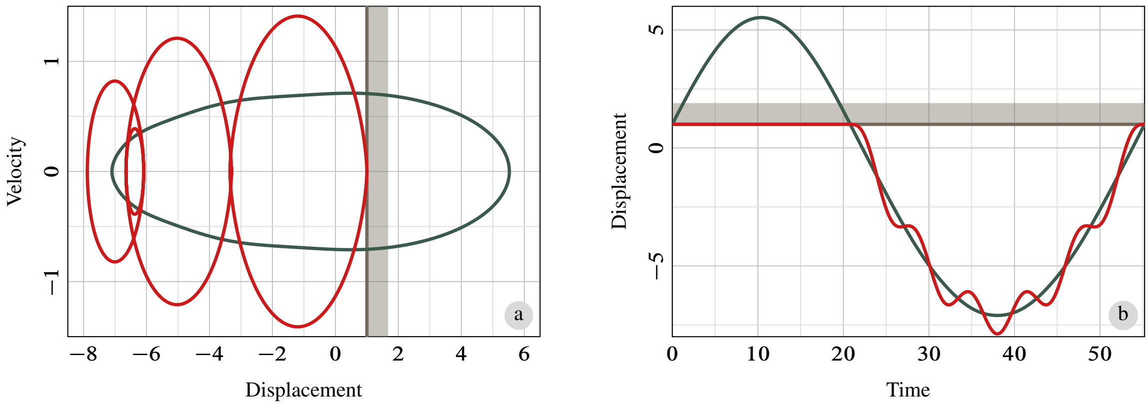

Figure 4: 1-SPP for $k_1 = k_2 = 1$ and $m_1 = 100$, $m_2 = 1$: $s \approx 34.412\,\mathrm{s}$ and $\mathcal{T} \approx 20.804\,\mathrm{s}$. (a) Orbits. (b) Displacements

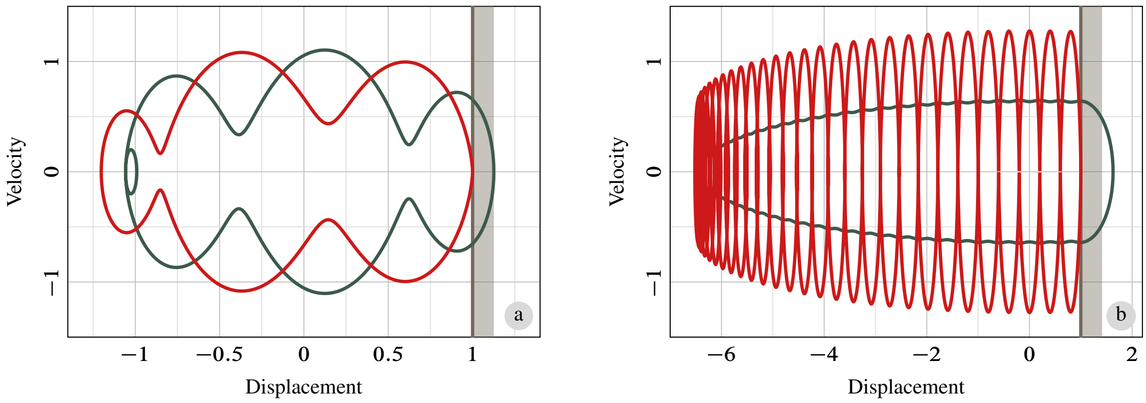

Figure 5: 1-SPP Orbits. (a) $k_2 = 10k_1$ and $m_1 = m_2$. (b) $k_2 = 100k_1$ and $m_1 = 100m_2$

large s, the free flight lasts a long period of time and the possibility that u_2 gets larger than d seems higher. Still, 1-SPP with large s are shown in Figures 4 and 5.

4. Sticking contact

This Section is devoted to the mathematical proof of Theorem 2.1 concerned with the necessary and sufficient conditions on the occurrence of a sticking phase (see also [7]). The theory on such systems with impacts can be found in [1, 3, 4, 6, 13].

To experience a sticking phase, a zero-velocity at the contact is the first necessary condition, see below or [3, 6]. Then, the sticking phase holds whenever there is a positive force generated from mass 1 acting on mass 2. This force is explicit through the last equation of (1.1a). Through Lemma 4.1, it is clear that the energy of the unconstrained linear free flight system is conserved by the sticking system below. The sticking system becomes simply a 1-dof problem, and the closed form as well as the explicit duration of sticking phase are obtained.

The loss of regularity induced by the sticking phase is also studied. The solution is very smooth away from the beginning and the end of the sticking phase, namely analytic [3]. The function u_2 belongs to the Sobolev space $W^{3,\infty}$ which means that $\ddot{u}_2$ is a Lipschitz function. The smoother function u_1 belongs to $C^4 \cap W^{5,\infty}$: for both functions, the singularity is located on the boundary of the sticking phase. The orbit $\{(u_2(t), \dot{u}_2(t)),\ t \in \mathbb{R}\}$ has only a $C^{1.5}$-regularity at the sticking point. This singularity $C^{1.5}$ is visible in Figure 2 and is caused by the zero velocity and zero acceleration of mass 2 exactly when the sticking phase starts and ends.

Proof. The right and left analyticity of the solution for the perfect elastic bounce is used (Proposition 19 in [3]). Note that the condition of a closed contact, *i.e.* $u_2(0) = d$, with zero velocity $\dot{u}_2^-(0) = 0$ is mandatory. Otherwise $\dot{u}_2^-(0) > 0$, $\dot{u}_2^+(0) = -\dot{u}_2^-(0) > 0$ and the mass immediately leaves the wall, that is $u_2(t) < d$ for $t \gtrsim 0$ such that there is no sticking phase. The second equation of System (1.1a) is rewritten with the aforementioned initial data for mass 2 only:

$$\begin{aligned} &m_2\ddot{u}_2(t) = k_2(u_1(t) - u_2(t)) + R(t) \\ &u_2(0) = d, \quad \dot{u}_2^-(0) = 0, \quad R(t) \leq 0. \end{aligned} \tag{4.1}$$

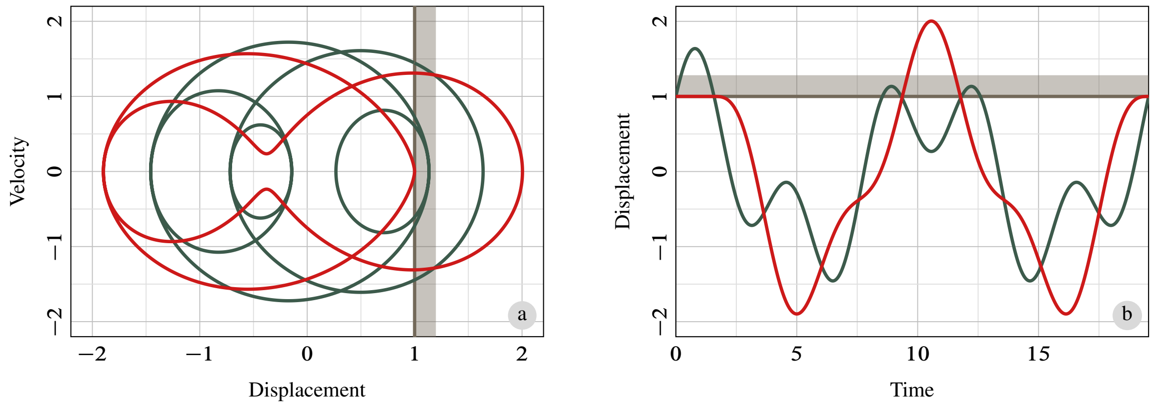

Figure 6: Non-admissible 1-SPP: mass 2 penetrates the wall during free flight. (a) Orbits. (b) Displacements

During a sticking phase, $u_2(t) \equiv d$ implies $\ddot{u}_2(t) \equiv 0$ and Equation (4.1) yields the relation

$$R(t) = k_2(d - u_1(t)) \tag{4.2}$$

which is non-positive if and only if $u_1(t) \geq d$. As a consequence the zero velocity $\dot{u}_2(0) = 0$ is not sufficient to ensure the existence of a sticking phase starting at time $t = 0$. Various situations depending on the state $(u_1(0), \dot{u}_1(0))$ should now be considered:

$u_1(0) < d$ — The left-hand side of (4.1) is negative. Thus $\ddot{u}_2^+(0) < 0$ and there is no sticking phase. More precisely, u_2 is a piecewise analytic function [3] and its Taylor series in the right neighbourhood of 0 is

$$u_2(t) = d + t^2 \frac{\ddot{u}_2^+(0)}{2} + \mathcal{O}(t^3) < d.$$

$u_1(0) > d$ — Since u_1 is continuous, it is larger than d in a right neighbourhood $[0\,;\varepsilon_2[$ of $t = 0$. Thus, there is a positive force $F(t) = k_2(u_1(t) - u_2(t))$ acting on mass 2, and by Newton's third law, there exists a reaction $R(t)$ such that $R(t) = -F(t)$. Substituting into (4.1) yields $\ddot{u}_2(t) = 0$, $\forall t \in [0\,;\varepsilon_2[$. Hence $u_2(t) = d$, $\forall t \in [0\,;\varepsilon_2[$, *i.e.* a sticking phase emerges.

$u_1(0) = d$ — $\ddot{u}_2^+(0) = 0$ and there are three possibilities for the velocity of mass 1:

1. If $\dot{u}_1(0) > 0$, $u_1(t)$ becomes immediately larger than d for $t > 0$ small enough. This is similar to the previous case where a sticking phase occurs.
2. If $\dot{u}_1(0) < 0$ then $u_1(t)$ becomes immediately smaller than d and no sticking phase occurs. More precisely from Equation (4.1), $m_2\dddot{u}_2^+(0) = k_2(\dot{u}_1(0) - \dot{u}_2(0)) < 0$ and the Taylor series of u_2 in the right neighbourhood of 0 is
$$u_2(t) = u_2(0) + t\dot{u}_2(0) + \frac{t^2}{2}\ddot{u}_2^+(0) + \frac{t^3}{6}\dddot{u}_2^+(0) + \mathcal{O}(t^4) = d + 0 + 0 + t^3\frac{\dddot{u}_2^+(0)}{6} + \mathcal{O}(t^4) < d.$$
3. If $\dot{u}_1(0) = 0$, then $\ddot{u}_1^+(0) = -k_1 d/m_1 < 0$. Thus $m_2\dddot{u}_2^+(0) = k_2(\dot{u}_1(0) - \dot{u}_2(0)) = 0$ and $m_2 u_2^{(4)+}(0) = k_2(\ddot{u}_1^+(0) - \ddot{u}_2^+(0)) < 0$. Similarly, a Taylor series of u_2 in the right neighbourhood of 0 shows that $u_2(t) < d$ for $t \gtrsim 0$ and there is no sticking phase.

Note that only the last case $u_1(0) = d$ and $\dot{u}_1(0) = 0$ crucially depends on the sign of d. It is further discussed in Section 7 when $d \leq 0$. Moreover, all piecewise analytic solutions presented above preserve energy. It is clear for the grazing case since $R \equiv 0$. When sticking occurs, energy conservation is a consequence of Lemma 4.1. In conclusion, every introduced case corresponds to the unique solution preserving energy [3]. ■

4.1. Sticking system From the previous developments, the sticking System [6] complemented by the initial data at the beginning of a sticking phase is explicitly derived as

$$m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2 u_2 = 0, \qquad u_1(0) = d, \qquad \dot{u}_1(0) = v > 0, \tag{4.3}$$

$$m_2\ddot{u}_2 = 0, \qquad u_2(0) = d, \qquad \dot{u}_2(0) = 0. \tag{4.4}$$

This system becomes simply an sticking equation below. The initial data for mass 1 has to be clarified. If $u_1(0) > d$ then this inequality is also valid locally in the past, and the sticking phase exists before $t = 0$. If $u_1(0) = d$ and $\dot{u}_1(0) > 0$, then $u_1(t) < d$ for $t \lesssim 0$ so there is no sticking phase just before $t = 0$, in other words, $t = 0$ is the beginning of the sticking phase. The grazing contact case $\dot{u}_1(0) = 0$ and the case where constraint (2.14) is violated, $\dot{u}_1(0) < 0$, do not have to be considered.

Notice that the system is not symmetric. However, the energy of the free flight system is conserved by the sticking system.

Lemma 4.1 *The solution of System* (4.3)-(4.4) *conserves the energy* $\mathbf{E}$ *expressed in* (1.1d).

Proof. Assume that the time $t = 0$ is the beginning of a sticking phase and $t = \tau$, the end. During this sticking phase on the interval $[0\,;\tau]$, the governing equations are

$$m_1\ddot{u}_1 + (k_1 + k_2)u_1 = k_2 d, \tag{4.5}$$

$$u_2 = d. \tag{4.6}$$

The first equation conserves the energy around the new equilibrium $\overline{u}_1 = k_2 d/(k_1 + k_2)$:

$$E_1(t) = m_1\dot{u}_1^2(t) + k_1(u_1(t) - \overline{u}_1)^2 = E_1(0).$$

Moreover, writing $u_1(t) = (u_1(t) - \overline{u}_1) + \overline{u}_1$ and since $u_2(t) = d$, an easy computation yields:

$$(k_1 + k_2)u_1^2(t) = (k_1 + k_2)(u_1(t) - \overline{u}_1)^2 + 2k_2 u_1(t)u_2(t) + C \quad \text{and} \quad C = 3(k_1 + k_2)\overline{u}_1^2.$$

The energy of System (1.1) can be computed. Notice that $\dot{\mathbf{u}}$ is continuous through a sticking phase so the exponent $\pm$ is dropped:

$$\begin{aligned}
\mathbf{E}(t) &= \dot{\mathbf{u}}^\top(t)\mathbf{M}\dot{\mathbf{u}}(t) + \mathbf{u}^\top(t)\mathbf{K}\mathbf{u}(t) \\
&= m_1\dot{u}_1^2(t) + (k_1 + k_2)u_1^2(t) + m_2\dot{u}_2^2(t) + k_2u_2^2(t) - 2k_2u_1(t)u_2(t) \\
&= m_1\dot{u}_1^2(t) + (k_1 + k_2)(u_1(t) - \overline{u}_1)^2 + 2k_2u_1(t)u_2(t) + C + 0 + k_2d^2 - 2k_2u_1(t)u_2(t) \\
&= E_1(t) + C + k_2d^2 = E_1(0) + C + k_2d^2 = \mathbf{E}(0).
\end{aligned}$$

■

The sticking system is now solved and the sticking time is explicitly exhibited: this is an interesting feature of the 2-dof mechanical system. The 1-dof linear oscillator problem with a constant force (4.5) has the explicit solution

$$u_1(t) = A\cos(\omega t + \phi) + \frac{k_2}{k_1 + k_2}d \quad \text{where} \quad \omega = \sqrt{\frac{k_1 + k_2}{m_1}}.$$

The expression of the constants A and ϕ stems from the initial condition $[u_1(0), \dot{u}_1(0)]^\top = [d, v]^\top$ as follows

$$A = \frac{k_1 d}{(k_1 + k_2)\cos(\phi)} \quad \text{and} \quad \phi = -\arctan(\xi v) \quad \text{with} \quad \xi = \frac{\sqrt{(k_1 + k_2)m_1}}{k_1 d}$$

and $\mathcal{T}$ is the first positive time satisfying $u_1(\mathcal{T}) = d$, that is $\mathcal{T} = 2\arctan(\xi v)/\omega$. Due to the symmetry of the solution to Problem (4.5) with respect to the u_1 axis in the plane $(u_1, \dot{u}_1)$, $u_1(\mathcal{T}) = d$ and $\dot{u}_1(\mathcal{T}) = -v$ which also means, through Theorem 2.1, that $\mathcal{T}$ is the end of the sticking phase.

4.2. 1.5-singularity at the sticking point The following Proposition states precisely the regularity near a sticking phase, essentially C^2 and almost C^3. The lower $C^{1.5}$-regularity of the orbit is obtained at the end of the Section.

Proposition 4.2 [Regularity of solutions] *Assume* $\mathbf{u}(\cdot)$ *is a solution of System* (1.1) *on* $[T_0\,; T_1]$ *with only a sticking phase on* $[0\,;\mathcal{T}]$ *and a free flight elsewhere with* $T_0 < 0 < \mathcal{T} < T_1$. *Then* $u_1 \in C^4([T_0\,; T_1]) \cap W^{5,\infty}([T_0\,; T_1])$ *and* $u_2 \in C^2([T_0\,; T_1]) \cap W^{3,\infty}([T_0\,; T_1])$.

Proof. Away from the sticking phase beginning and end, the solution is regular: analytic outside $[0\,;\mathcal{T}]$, constant inside $]0\,;\mathcal{T}[$. The regularity at $t = 0$ and $t = \mathcal{T}$ is of higher interest. Only the case $t = 0$ is considered since its counterpart at $t = \mathcal{T}$ is similar in nature. The initial data at $t = 0$ are $[\mathbf{u}(0), \dot{\mathbf{u}}(0)]^\top = [d, d, v, 0]^\top$. The second Equation of (1.1a) is

$$m_2\ddot{u}_2(t) = k_2(u_1(t) - u_2(t)) + R(t). \tag{4.7}$$

During the sticking phase $0 < t < \mathcal{T}$, $u_2(t) = d$ so $\ddot{u}_2(t) = 0$ and $\ddot{u}_2^+(0) = 0$. Before the sticking phase $t < 0$, $R(t) = 0$ since $u_2(t) < d$ and $\lim_{0>t\to 0} u_2(t) = d = \lim_{0>t\to 0} u_1(t)$ and from Equation (4.7), $\ddot{u}_2^-(0) = 0$, thus $\ddot{u}_2$ is continuous at time $t = 0$ with $\ddot{u}_2(0) = 0$. However, the third derivative of u_2 on the left of $t = 0$

does not vanish since $m_2\ddot{u}_2^-(0) = k_2(\dot{u}_1(0) - \dot{u}_2(0)) = k_2 v > 0$ which also means that $\ddot{u}_2$ is bounded. Hence, $u_2 \in C^2([0;T]) \cap W^{3,\infty}([0;T])$.

The regularity of u_1 is investigated from the first Equation of (1.1a) $m_1\ddot{u}_1 + (k_1 + k_2)u_1 = k_2 u_2$ showing that $\ddot{u}_1$ and u_2 have the same regularity. Accordingly, $\ddot{u}_1 \in C^2([0;T]) \cap W^{3,\infty}([0;T])$ that is $u_1 \in C^4([0;T]) \cap W^{5,\infty}([0;T])$. ■

We now prove the $C^{1.5}$-regularity of the orbit without using any explicit formula.

Proof. The $C^{1.5}$-smoothness of the projection of the orbit on the last component, more precisely the regularity of the set $\Gamma_2 = \{\gamma(t) = (u_2(t), \dot{u}_2(t)),\ 0 \le t \le T\} \subset \mathbb{R}^2$ is explored. By T-periodicity, this parametrization is defined for all time. During the sticking phase $0 \le t \le \mathcal{T}$, the last mass rests against the foundation, that is $\gamma(t) = \gamma(0) = (d, 0)$ and $\dot{\gamma}(t) = (0, 0)$: the parametrization is then singular. Instead, a regular parametrization of Γ_2 is proposed as

$$\tilde{\gamma}(t) = \gamma(t - \mathcal{T}), \quad \mathcal{T} \le t \le T. \tag{4.8}$$

In other words, $\tilde{\gamma}$ is γ where the sticking phase has been removed and is also defined for all time through s-periodicity with $s = T - \mathcal{T}$. The set $\tilde{\Gamma}_2 = \tilde{\gamma}([0;s])$ is exactly Γ_2. The curve is analytic except at the sticking point $(d, 0)$. A precise study of $\tilde{\gamma}(t)$, $|t| < \varepsilon$ should now be undertaken for $\varepsilon > 0$ sufficiently small. To this end, the left and right derivatives are computed since the solution is left and right analytic at the sticking point [3]:

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\tilde{\gamma}^-(0) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\gamma^-(0), \quad \frac{\mathrm{d}^k}{\mathrm{d}t^k}\tilde{\gamma}^+(0) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\gamma^+(\mathcal{T}). \tag{4.9}$$

To compute the successive left and right derivatives, the ODE $m_2\ddot{u}_2(t) = k_2(u_1(t) - u_2(t))$ is used just before the sticking phase and just after the sticking phase. Recall that $u_1(0) = d$ and $\dot{u}_1(0) = v > 0$, $u_2(0) = d$ and $\dot{u}_2(0) = 0$, $u_1(\mathcal{T}) = d$ and $\dot{u}_1(\mathcal{T}) = -v < 0$, $u_2(\mathcal{T}) = d$ and $\dot{u}_2(\mathcal{T}) = 0$. The ODE gives $m_2\ddot{u}_2^-(0) = k_2(d - d) = 0$, $\ddot{u}_2^+(\mathcal{T}) = 0$, so $\dot{\tilde{\gamma}}^\pm(0) = (0, 0)$. The parametrization is still singular and higher derivatives of u_2 are computed by differentiating the ODE:

$$\begin{aligned}
&m_2\dddot{u}_2(t) = k_2(\dot{u}_1(t) - \dot{u}_2(t)) \\
&\ddot{\tilde{\gamma}}^-(0) = (0, \beta), \quad \ddot{\tilde{\gamma}}^+(0) = (0, -\beta), \quad \beta = k_2 v/m_2 > 0 \\
&m_2\ddddot{u}_2(t) = k_2(\ddot{u}_1(t) - \ddot{u}_2(t)) \\
&\dddot{\tilde{\gamma}}^-(0) = (\beta, \delta), \quad \dddot{\tilde{\gamma}}^+(0) = (-\beta, \delta), \quad \delta = -k_2 k_1 d/(m_2 m_1) < 0
\end{aligned}$$

where the second derivative of u_1 comes from the equation $m_1\ddot{u}_1(t) = -k_1 u_1(t) - k_2(u_1(t) - u_2(t))$: $\ddot{u}_1(0) = -k_1 d/m_1 < 0$. The local behaviour at $t = 0$ is then for $\pm t > 0$:

$$\tilde{\gamma}(t) = (d, 0) + \frac{1}{2}\operatorname{sign}(t)(0, \beta)t^2 + \frac{1}{6}(\operatorname{sign}(t)\beta, \delta)t^3 + \mathcal{O}(t^4). \tag{4.10}$$

There are two singularities for this parametrization: the left and right expansions for $\pm t > 0$, and the more important $\dot{\tilde{\gamma}}(0) = (0, 0)$. To clearly identify the regularity of the curve at $t = 0$, a last change of variable is performed [12]: $\tau = \operatorname{sign}(t)t^2$ and $\hat{\gamma}(\tau) = \tilde{\gamma}(t)$ such that:

$$\hat{\gamma}(\tau) = (d, 0) + \frac{1}{2}(0, \beta)\tau + \frac{1}{6}(\beta, \operatorname{sign}(\tau)\delta)|\tau|^{1.5} + \mathcal{O}(\tau^2). \tag{4.11}$$

The $C^{1.5}$-regularity is then identified since $\dot{\hat{\gamma}}(0) \ne (0, 0)$ and this is optimal. ■

5. Building 1-SPP

This Section addresses the construction of the 1-SPP developed in Section 2.2. An explicit formula for $\mathcal{T}$ is obtained and the set of admissible initial data is derived. The initial velocity of the first mass depends on the free flight time s and it is proven that s can be found in the infinite set of roots of $h(s)$. The symmetry of the solutions is discussed.

5.1. Initial data

Without loss of generality, the initial data are defined at the Poincaré section $u_2 = d$. The problem is to find a periodic function $\mathbf{u}$ associated with the initial data $[d, d, v, 0]^\top$ such that there is one sticking phase per period. As explained previously, T and $\mathcal{T}$ are parametrized by s. The sticking solution and the sticking time $\mathcal{T} > 0$ are calculated explicitly in Section 4.1.

By denoting $\mathbf{U} = [\mathbf{u}, \dot{\mathbf{u}}]^\top$, a free flight starts at time $\mathcal{T}$ with the initial data

$$\mathbf{U}(\mathcal{T}) = [d, d, -v, 0]^\top. \tag{5.1}$$

It can be written as

$$\mathbf{U}(\mathcal{T}) = \mathbf{S}\mathbf{U}(0) \quad \text{where} \quad \mathbf{S} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.2}$$

After the sticking phase, System (1.1a) becomes

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}. \tag{5.3}$$

We shall find the solution $\mathbf{u}$ such that

$$\mathbf{U}(T) = \mathbf{U}(0). \tag{5.4}$$

Through the change of variable $\mathbf{u} = \mathbf{P}\mathbf{q}$, Equation (5.3) becomes $\mathbf{I}\ddot{\mathbf{q}} + \mathbf{\Omega}^2\mathbf{q} = \mathbf{0}$ which features the following block matrix solution

$$\mathbf{Q}(t) = \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} = \mathbf{R}(t - \mathcal{T}) \begin{bmatrix} \mathbf{q}(\mathcal{T}) \\ \dot{\mathbf{q}}(\mathcal{T}) \end{bmatrix}, \quad \forall t \in]\mathcal{T}; T[\tag{5.5}$$

where

$$\mathbf{R}(t) = \begin{bmatrix} \mathbf{cos}(t\mathbf{\Omega}) & \mathbf{\Omega}^{-1}\,\mathbf{sin}(t\mathbf{\Omega}) \\ -\mathbf{\Omega}\,\mathbf{sin}(t\mathbf{\Omega}) & \mathbf{cos}(t\mathbf{\Omega}) \end{bmatrix}. \tag{5.6}$$

Using Equation (5.4), the period T of the 1-SPP satisfies

$$\mathbf{Q}(T) = \mathbf{Q}(0). \tag{5.7}$$

Denote $s = T - \mathcal{T}$, Equation (5.7) projected onto modal coordinates reads

$$\mathbf{R}(s)\tilde{\mathbf{S}}\mathbf{Q}(0) = \mathbf{Q}(0) \quad \text{where} \quad \tilde{\mathbf{S}} = \left[\mathbf{P}^{-1}\right]\mathbf{S}\,[\mathbf{P}] \quad \text{and} \quad [\mathbf{P}] = \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{bmatrix}$$

which can be expressed as $(\mathbf{R}(s)\tilde{\mathbf{S}} - \mathbf{I})\mathbf{Q}(0) = \mathbf{0}$ with

$$\mathbf{R}(s)\tilde{\mathbf{S}} - \mathbf{I} = \begin{bmatrix} \mathbf{cos}(s\mathbf{\Omega}) - \mathbf{I} & \mathbf{\Omega}^{-1}\,\mathbf{sin}(s\mathbf{\Omega})\mathbf{P}^{-1}\mathbf{L}\mathbf{P} \\ -\mathbf{\Omega}\,\mathbf{sin}(s\mathbf{\Omega}) & \mathbf{cos}(s\mathbf{\Omega})\mathbf{P}^{-1}\mathbf{L}\mathbf{P} - \mathbf{I} \end{bmatrix}.$$

The computations are similar to those introduced in [8]. This similarity will be explained later through the relationship between 1-SPP and the one-Impact-Per-Period solutions (1-IPP) detailed in [8][2].

Assume that s is not a period of the linear differential system, $s \notin \cup_{j=1}^{2} T_j\mathbb{Z}$ where $T_j = 2\pi/\omega_j$, $j = 1, 2$ are the natural periods of the underlying linear system. Then, the following quantities are well defined:

$$\mathbf{\Phi}(s) = (\mathbf{I} - \mathbf{cos}(s\mathbf{\Omega}))^{-1}\mathbf{\Omega}^{-1}\,\mathbf{sin}(s\mathbf{\Omega}), \tag{5.8}$$

$$\mathbf{w}(s) = \mathbf{P}\mathbf{\Phi}(s)\mathbf{P}^{-1}\mathbf{L}\mathbf{e}_1, \quad \mathbf{e}_1 = (1, 0)^\top, \tag{5.9}$$

$$w_1(s) = \mathbf{e}_1^\top\mathbf{w}(s). \tag{5.10}$$

The set of possible initial data, yielding 1-SPP, or "ghosts" if Constraint (2.14) is not satisfied, is described explicitly through the following lemma:

Lemma 5.1 *If $s \notin \cup_{j=1}^{2} T_j\mathbb{Z}$ then the system*

$$\mathbf{R}(s)\tilde{\mathbf{S}}\mathbf{Q}(0) = \mathbf{Q}(0) \tag{5.11}$$

defines a one dimensional vector space parametrized by $c \in \mathbb{R}$ given in variables

$$\begin{bmatrix} \mathbf{u}(0) \\ \dot{\mathbf{u}}(0) \end{bmatrix} = \mathbf{P} \begin{bmatrix} \mathbf{q}(0) \\ \dot{\mathbf{q}}(0) \end{bmatrix} = c \begin{bmatrix} \mathbf{w}(s) \\ \mathbf{e}_1 \end{bmatrix}. \tag{5.12}$$

Proof. Compute $\ker(\mathbf{R}(s)\tilde{\mathbf{S}} - \mathbf{I})$ by blocks (see [8]):

$$\begin{bmatrix} \mathbf{cos}(s\mathbf{\Omega}) - \mathbf{I} & \mathbf{\Omega}^{-1}\,\mathbf{sin}(s\mathbf{\Omega})\mathbf{P}^{-1}\mathbf{L}\mathbf{P} \\ -\mathbf{\Omega}\,\mathbf{sin}(s\mathbf{\Omega}) & \mathbf{cos}(s\mathbf{\Omega})\mathbf{P}^{-1}\mathbf{L}\mathbf{P} - \mathbf{I} \end{bmatrix} \sim \begin{bmatrix} \mathbf{cos}(s\mathbf{\Omega}) - \mathbf{I} & \mathbf{\Omega}^{-1}\,\mathbf{sin}(s\mathbf{\Omega})\mathbf{P}^{-1}\mathbf{L}\mathbf{P} \\ \mathbf{0} & (\mathbf{L} + \mathbf{I})\mathbf{P} \end{bmatrix} \tag{5.13}$$

because the matrix $(\mathbf{I} - \mathbf{cos}(s\mathbf{\Omega}))^{-1}\mathbf{P}^{-1}$ is invertible. Since $\dot{\mathbf{u}} = \mathbf{P}\dot{\mathbf{q}}$, the right lower blocks in (5.13) simplifies to $(\mathbf{L} + \mathbf{I})\dot{\mathbf{u}} = \mathbf{0}$, that is $\dot{\mathbf{u}} = c\mathbf{e}_1$ with $c \in \mathbb{R}$. Similarly, the upper block provides the expression $\mathbf{q} = c(\mathbf{I} - \mathbf{cos}(s\mathbf{\Omega}))^{-1}\mathbf{\Omega}^{-1}\,\mathbf{sin}(s\mathbf{\Omega})\mathbf{P}^{-1}\mathbf{L}\mathbf{e}_1$. ■

[2] Note that there is a change of sign in $\mathbf{w}(s)$ due to the coefficient $a_{kj} = -P_{kj}P_{j1}^{-1}$ instead of $P_{kj}P_{jN}^{-1}$ in [8].

The parameter c is identified from the third row of (5.12), $c = \dot{u}_1^+(0) = v$. By expressing the initial condition

$$\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} d \\ d \end{bmatrix} = v \begin{bmatrix} w_1(s) \\ w_2(s) \end{bmatrix} = c\, \mathbf{P\Phi}(s)\mathbf{P}^{-1}\mathbf{Le}_1 \tag{5.14}$$

System (5.14) simplifies to $w_1(s) = w_2(s)$ or

$$h(s) = w_1(s) - w_2(s) = \sum_{j=1}^{2} \alpha_j \cot\left(\frac{\omega_j s}{2}\right) = 0$$

and the initial velocity of the first mass is found from

$$v = d/w_1(s), \quad w_1(s) > 0. \tag{5.15}$$

If $\omega_1/\omega_2 \notin \mathbb{Q}$, the function $h(s)$ exhibits a countably infinite set of roots s. Moreover the set of s such that $h(s) = 0$ and $v(s) > 0$ is also countably infinite by Theorem 2.3. The particular case $\omega_1/\omega_2 \in \mathbb{Q}$ is discussed in Section 6.

5.2. Symmetry

To conclude the validation of Theorem 2.2, the symmetry of 1-SPP is proved.

Proof. Through periodicity, it is sufficient to check the symmetry of the solutions on one period. The symmetry is satisfied during the sticking phase and the free flight. Since only the first mass oscillates during the sticking phase, the solution is symmetric. Let us check the symmetry of solutions during the free flight time $t \in [\mathcal{T}\,;T]$ where $\mathbf{u}(\mathcal{T}) = \mathbf{u}(T)$ and $\dot{\mathbf{u}}^+(\mathcal{T}) = -\dot{\mathbf{u}}^-(T)$. Denoting $\theta = (T + \mathcal{T})/2$, it is sufficient to show that $\mathbf{u}(\theta + t) = \mathbf{u}(\theta - t)$, $\forall t \in I = [-s/2\,; s/2]$. Let $\mathbf{z}_+$ be the function defined on I such that $\mathbf{z}_+(t) = \mathbf{u}(\theta + t)$. Then $\mathbf{z}_+$ is a well defined smooth function on I with $\mathbf{z}_+(s/2) = \mathbf{u}(T)$ and $\dot{\mathbf{z}}_+(s/2) = \dot{\mathbf{u}}^-(T)$. Similarly, by defining the function $\mathbf{z}_-(t) := \mathbf{u}(\theta - t)$, for $t \in I$, it can be checked that $\mathbf{z}_-(s/2) = \mathbf{u}(\mathcal{T})$ and $\dot{\mathbf{z}}_-(s/2) = -\dot{\mathbf{u}}^+(\mathcal{T})$. Furthermore, both $\mathbf{z}_+$ and $\mathbf{z}_-$ are solutions to the linear differential system $\mathbf{M}\ddot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \mathbf{0}$ on I. Notice that $\mathbf{z}_+$ and $\mathbf{z}_-$ have the same initial data $\mathbf{z}_+(s/2) = \mathbf{z}_-(s/2)$ and $\dot{\mathbf{z}}_+(s/2) = \dot{\mathbf{z}}_-(s/2)$. Hence, by the uniqueness of the initial value problem, it is deduced that $\mathbf{z}_+(t) \equiv \mathbf{z}_-(t)$ on I. ■

5.3. Relationship between 1-SPP and 1-IPP

For this two-degree-of-freedom vibro-impact system, a relationship between one-sticking-phase-per-period and one-impact-per-period solutions [8] is exhibited. It clarifies the similarities and differences of such periodic solutions.

Consider the two Figures 2(a) and 2(b) showing a single loop in Γ_2. Figures 7(a) and 7(b) are then obtained by "deleting" the sticking phase on the whole interval $]0\,; \mathcal{T}[$ such that a 1-IPP solution [7, 8] is identified, where the jump occurs on mass 1 (instead of mass 2) when $u_1(0) = d$ as well as $u_2(0) = d$. It is important to note that this 1-IPP is "unique" in the sense that it satisfies $u_2(0) = d$; it is denoted 1-IPP$_\text{p}$ in the remainder.

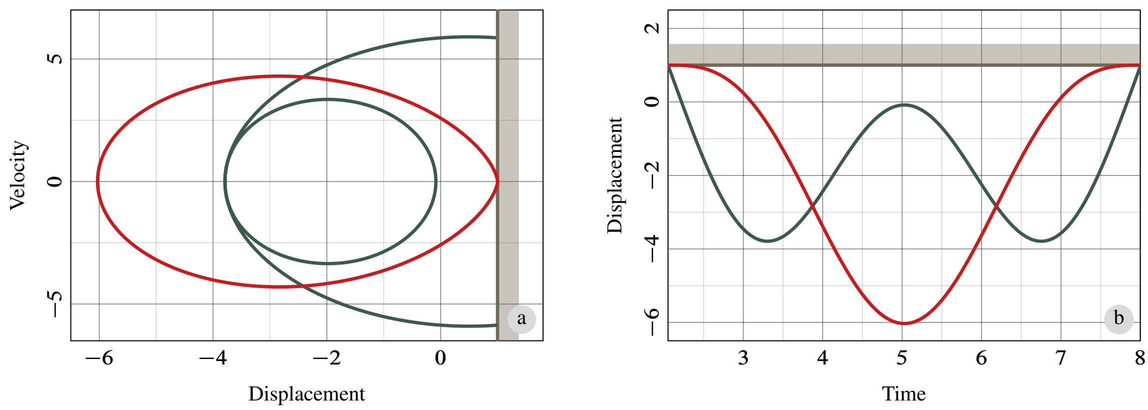

Figure 7: Is a 1-IPP or a 1-SPP without sticking phase drawn? (a) Orbits. (b) Displacements

The correspondence between 1-SPP and this particular 1-IPP$_\text{p}$ is now detailed. To this end, generalized 1-SPP and 1-IPP$_\text{p}$, *ie* G1-SPP and G1-IPP$_\text{p}$ respectively, are first defined: they are 1-SPP and 1-IPP$_\text{p}$ unconstrained during the free flight and u_1 as well as u_2 might exceed d[3]. By definition, a G1-SPP $\mathbf{u}$

[3] The G1-IPP in this paper has a counterpart in [8]: it is a G1-IPP where the jump in velocity affects the second mass instead of the first mass here. As such, we know that there is a unique G1-IPP for all positive periods. This is the reason why the formulas

satisfies the following requirements:

1. $s \in Z$,
2. $T = s + \mathcal{T}$ is the fundamental period with $\mathcal{T} = \mathcal{T}(s)$,
3. a sticking phase on $]0\,;\mathcal{T}(s)[$ with $u_1(0) = d$, $\dot{u}_1(0) = v = v(s)$, $u_2(0) = d$, $\dot{u}_2(0) = 0$,
4. a free flight on $]\mathcal{T}\,; T[$ with $u_1(\mathcal{T}) = d$, $\dot{u}_1(\mathcal{T}) = -v$, $u_2(\mathcal{T}) = d$, $\dot{u}_2(\mathcal{T}) = 0$, $\mathbf{M\ddot{u}} + \mathbf{Ku} = \mathbf{0}$.

The one-to-one correspondence from Figure 2 to Figure 7 is formalized as

$$\widetilde{u}(t) = \mathbf{u}(\mathcal{T} + t), \quad 0 < t < s, \tag{5.16}$$

where $\widetilde{u}$ is taken s-periodic so that $\widetilde{u}^-(0) = \mathbf{u}(0)$, $\widetilde{u}^+(0) = \mathbf{u}(\mathcal{T})$. As a consequence, $\widetilde{u}$ is such that

1. s is the fundamental period,
2. $\widetilde{u}_1^{\pm}(0) = d$, $\dot{\widetilde{u}}_1^-(0) = \dot{u}_1(0) = v$, $\dot{\widetilde{u}}_1^+(0) = \dot{u}_1(\mathcal{T}) = -v$,
3. $\widetilde{u}_2^{\pm}(0) = d$, $\dot{\widetilde{u}}_2^{\pm}(0) = 0$,
4. a free flight on $]0\,; s[$: $\mathbf{M}\ddot{\widetilde{u}} + \mathbf{K}\widetilde{u} = \mathbf{0}$.

We can check that $\tilde{\mathbf{u}}$ is a G1-IPP$_\text{p}$. The only surprising condition is $\dot{\widetilde{u}}_2(0) = 0$ but zero velocity is automatically achieved by a G1-IPP [3, 8].

Proposition 5.2 [G1-SPP $\Leftrightarrow$ G1-IPP$_\text{p}$] *There is a one-to-one correspondence between G1-SPP with a sticking phase for mass 2 and G1-IPP$_p$.*

> ***Proof.*** This is a brief sketch. G1-SPP $\Rightarrow$ G1-IPP$_\text{p}$ was explained previously. Conversely, from a given G1-IPP$_\text{p}$, it is possible to build a sticking phase as in the proof of Theorem 2.1 in Section 4 with a free flight duration s to then define a unique G1-SPP. ■

The key parameter s appears to be simply the period of the associated G1-IPP$_\text{p}$. This proposition shows that the set Z corresponds exactly to the set of all G1-IPP$_\text{p}$. Let us state briefly the correspondence between Z^+ and Z^{ad} and the corresponding subset of all G1-IPP$_\text{p}$.

Concerning generalized solutions with a positive velocity at the impact ($v > 0$), it can be said that for all $s \in Z^+$, there exists a unique G1-SPP and a corresponding unique G1-IPP$_\text{p}$ which has a physical initial data at the impact time (no violation of the constraint near the impact time). Conversely, if a G1-IPP$_\text{p}$ is such that, at the impact time, the incoming velocity of mass 1 is positive then the period belongs to Z^+ which corresponds to a unique G1-SPP.

Finally, a 1-SPP, *i.e.* a G1-SPP satisfying the constraint $u_2(t) \leq d$ for all time, is in a unequivocal correspondence with a G1-IPP$_\text{p}$ satisfying the same constraint. Notice that is not the constraint to be a 1-IPP since the constraint for 1-IPP is on mass 1. Figures 2 and 7 show a perfect and rare correspondence between a 1-SPP and a 1-IPP since the associated G1-IPP$_\text{p}$ satisfies the two constraints $u_k(t) \leq d$ for all time and $k = 1, 2$. As a consequence, 1-SPP are isolated solutions. The reason lies in the fact that the space of G1-IPP is a one-dimensional manifold which intersects $\widetilde{u}_2 = d$ on a discrete set such that the G1-IPP become isolated. Another possible consequence, which is not further discussed here, is the existence of 1-SPP (if we are able to obtain such particular G1-IPP).

6. The countable set $Z^{\pm}$

The set $Z^{\pm}$ plays a key role to find admissible solutions, or more precisely admissible initial data which can satisfy the constraint $u_2 \leq d$ at least locally near the sticking phase. Thus, a question of interest emerges: what is the size of this set? In this Section, the sets Z^+ and Z^- are proven to be countably infinite if some generic assumptions are fulfilled. The proof of Theorem 2.3 is similar for both sets and only the proof for Z^+ is presented.

6.1. $Z^{\pm}$ is infinite when no resonance

Before stating the main proof with $\omega_1/\omega_2 \notin \mathbb{Q}$, we start with Lemma 6.1 below. Denote the orbit $\mathcal{O}$ in the torus $\Pi = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$:

$$\mathcal{O} = \{(x, y) = (\bar{t}, \overline{\rho t}) | t > 0\} \tag{6.1}$$

where $\bar{t} = t + 2\pi\mathbb{Z}$ and ρ, a constant.

Lemma 6.1 [Transversality and density] *Let f be a $2\pi-$periodic continuously differentiable function from $[0\,; 2\pi[$ to $[0\,; 2\pi[$. For any irrational number ρ, if $(x_0, y_0 = f(x_0))$ located on the curve $\mathcal{C}$ defined*

in Section 5 are slightly different from [8]. The condition on mass 1 in Theorem 2.1 corresponds to an elastic impact for mass 1.

by the graph of f satisfies the transversal condition between $\mathcal{C}$ and $\mathcal{O}$, that is

$$\dot{f}(x_0) \neq \rho \tag{6.2}$$

then $\forall \varepsilon > 0$, $\exists t > 0$ such that $\overline{\rho t} = f(\bar{t})$ and $|\bar{t} - x_0| < \varepsilon$.

In other words, every point on the curve $\mathcal{C}$ at which the tangent is transverse to the orbit $\mathcal{O}$ is an accumulation point of $\mathcal{O} \cap \mathcal{C}$, see Figure 8. Precisely, the set $\mathcal{O} \cap \mathcal{C}$ is dense in $\{(x, f(x)) | \dot{f}(x) \neq \rho\}$.

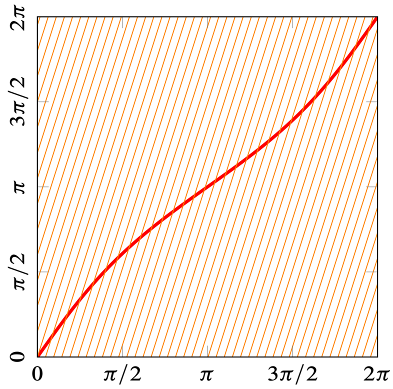

Figure 8: Density of $\mathbf{O} \cap \mathbf{H}$ in $\mathbf{H}$

Moreover, for all $A > 0$, the set $\mathcal{O}_A = \{(x, y) = (\bar{t}, \overline{\rho t}) | t > A\}$ shares the same property.

Proof. Assume $\rho > 0$, the cases $\rho = 0$ and $\rho < 0$ follow immediately.

Since $\dot{f}(x_0) \neq \rho$ and $\dot{f}$ is continuous, there exists $\varepsilon_0 > 0$ small enough such that $\dot{f}(x) \neq \rho$ $\forall x \in [x_0 - \varepsilon_0 \,; x_0 + \varepsilon_0]$. Without loss of generality, assume that $\dot{f}(x) > \rho$, $\forall x \in [x_0 - \varepsilon_0 \,; x_0 + \varepsilon_0]$. Since $\mathcal{O}$ is dense in Π, $\forall \varepsilon > 0$, there exists $t_0 > 0$ such that $z = (\overline{t_0}, \overline{\rho t_0})$ belongs to $\mathcal{O}$ close enough to (x_0, y_0), *i.e.* $|\overline{t_0} - x_0| < \varepsilon$ and $|\overline{\rho t_0} - y_0| < \varepsilon$: if z is on the curve $\mathcal{C}$ then t is chosen to be t_0, else z is above the curve $\mathcal{C}$, *i.e.* $\overline{\rho t_0} > f(\overline{t_0})$.

We will show that the orbit $\mathcal{O}$ intersects the curve $\mathcal{C}$ inside the box $]x_0 - \varepsilon_0 \,; x_0 + \varepsilon_0[\times]y_0 - k\varepsilon_0 \,; y_0 + k\varepsilon_0[$ where k is the maximum of $|\dot{f}|$ on $[x_0 - \varepsilon_0 \,; x_0 + \varepsilon_0]$ as shown in Figure 9. For this purpose, we use a line

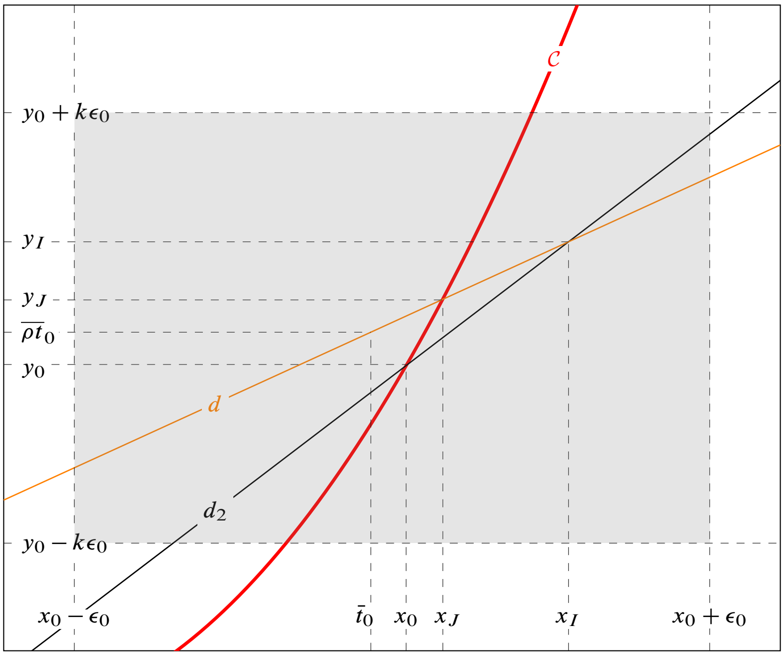

Figure 9: Zoom in the box $]x_0 - \varepsilon_0 \,; x_0 + \varepsilon_0[\times]y_0 - k\varepsilon_0 \,; y_0 + k\varepsilon_0[$ when $\dot{f}(x_0) > \rho$

d_2 under the curve on the right of (x_0, y_0). From $p = \min_{[x_0 - \varepsilon_0 \,; x_0 + \varepsilon_0]} \dot{f}$, the equation of the line d_2 with slope p passing through (x_0, y_0) is $y = p(x - x_0) + y_0$. The line d with slope ρ passing through $(\overline{t_0}, \overline{\rho t_0})$ and associated to the orbit $\mathcal{O}$ is defined by $y = \rho(x - \overline{t_0}) + \overline{\rho t_0}$. Let $I(x_I, y_I)$ be the intersection of those two lines. Since $p > \rho$, we have

$$x_I = \frac{px_0 - y_0 - \rho\overline{t_0} + \overline{\rho t_0}}{p - \rho}.$$

Choosing ε small enough such that $\varepsilon < \varepsilon_0(|p-\rho|)/(\rho+k)$ implies $x_I \in [x_0 - \varepsilon_0\,; x_0 + \varepsilon_0]$.

Consider the two curves d_2 and $\mathcal{C}$ intersecting at (x_0, y_0) and satisfying $\dot{f}(x) > p$ for all $x \in [x_0 - \varepsilon_0\,; x_0 + \varepsilon_0]$. Since $p > \rho$, d intersects d_2 at I. Hence, there exists an intersection of $\mathcal{C}$ and d in the interval $]x_0\,; x_I[$. In other words, there exists $t > 0$ such that $\overline{\rho t} = f(\bar{t})$ and $|\bar{t} - x_0| < \varepsilon_0$. The proof when z is under the curve $\mathcal{C}$ is similar. ■

The proof of Theorem 2.3 starts by showing that the set $Z = \{s > 0,\ h(s) = 0\}$ is countably infinite. It is true for the set $\{(\omega_1 s, \omega_2 s),\ h(s) = 0\}$ and will be useful to prove that the set Z^+ of free flight times s with admissible initial velocity $v(s) > 0$ is also countably infinite.

Proof. Set $\varphi(t) = \cot(t/2)$, then $h(s) = \alpha_1\varphi(\omega_1 s) + \alpha_2\varphi(\omega_2 s)$ and $w_1(s) = \beta_1\varphi(\omega_1 s) + \beta_2\varphi(\omega_2 s)$ where $\beta_j = b_{1j}$, and α_j, b_{kj} are defined in Equation (2.8). For every $(x, y) \in \Pi = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$, the two functions $H(x, y) = \alpha_1\varphi(x) + \alpha_2\varphi(y)$ and $W(x, y) = \beta_1\varphi(x) + \beta_2\varphi(y)$ correspond to $h(s) = H(\omega_1 s, \omega_2 s)$ and $w_1(s) = W(\omega_1 s, \omega_2 s)$. In order to simplify, the sets $\mathbf{O} = \{(\overline{\omega_1 s}, \overline{\omega_2 s}) | s > 0\}$, $\mathbf{H} = \{(x, y) \in \Pi | H(x, y) = 0\}$, and $\mathbf{W} = \{(x, y) \in \Pi | W(x, y) = 0\}$ are defined on the torus Π; $\mathbf{W}^+$, $\mathbf{W}^-$ are denoted as the domains of Π where $W(x, y) > 0$ and < 0, respectively.

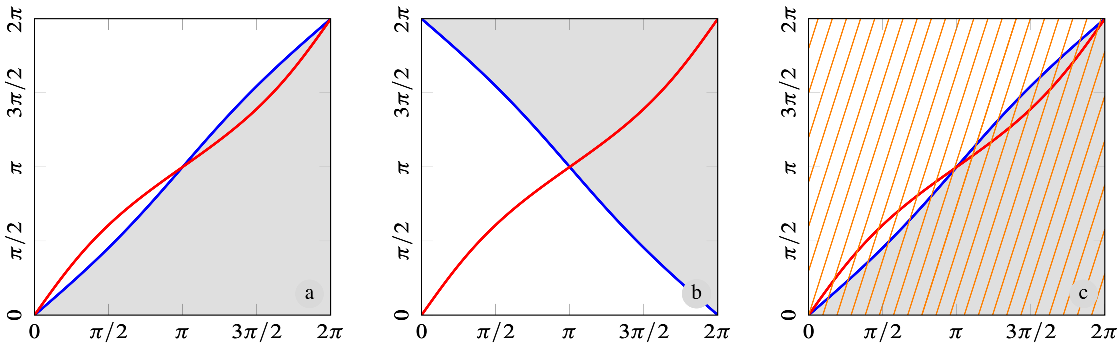

Figure 10: (a) or (b) $\mathbf{H} \cap \mathbf{W}^+$ is the half of the red curve which lies in the grey domain; (c) $\mathbf{O} \cap \mathbf{H} \cap \mathbf{W}^+$ is the set of all intersections between the red curve and the orange lines within the grey domain

The set $\mathbf{O}$ is equal to $\mathcal{O}$ with $\rho = \omega_2/\omega_1$. Consider the map $\gamma : \mathbb{R}^+ \to \mathbf{O}$, $s \mapsto (\overline{\omega_1 s}, \overline{\omega_2 s})$, then γ is bijective for $\omega_2/\omega_1 \notin \mathbb{Q}$ and

$$\gamma(Z) = \mathbf{O} \cap \mathbf{H} \tag{6.3}$$

$$\gamma(Z^+) = \mathbf{O} \cap \mathbf{H} \cap \mathbf{W}^+ \tag{6.4}$$

Hence, instead of proving the set Z is countably infinite, the stronger result $\overline{\mathbf{O} \cap \mathbf{H}} = \mathbf{H}$ is proven. This implies $\mathbf{O} \cap \mathbf{H}$ is countably infinite. This stronger result shows that Z^+ is countably infinite by pointing out the density of $\mathbf{O} \cap \mathbf{H} \cap \mathbf{W}^+$ in $\mathbf{H} \cap \mathbf{W}^+$ and the countable infinity of $\mathbf{H} \cap \mathbf{W}^+$.

To show that $\overline{\mathbf{O} \cap \mathbf{H}} = \mathbf{H}$, assume $\alpha_2 \neq 0$, rewrite $H(x, y) = 0$ to have $y = \psi(x)$ where $\psi = \varphi^{-1}(r\varphi)$ and $r = -\alpha_1/\alpha_2$.

1. We show that $\dot{\psi} \neq \rho$ almost everywhere. Since ψ is an analytic function on $I =]0\,; 2\pi[$, so is $\dot{\psi}$. After simplification, the derivative of ψ becomes

$$\dot{\psi} = \frac{r(1+\varphi^2)}{1+r^2\varphi^2} = \frac{1}{r}\left(1 + \frac{r^2-1}{1+r^2\varphi^2}\right) \tag{6.5}$$

   which degenerates to a constant function for $r = \pm 1$. Otherwise, $\dot{\psi}$ is not a constant function and the set $\{x \in I | \dot{\psi}(x) = \rho\}$ is empty or countable. Hence, $\dot{\psi} \neq \rho$ holds almost everywhere. It is still true if $\alpha_2 = 0$ since $H(x, y)$ becomes a periodic function of x, and $\mathbf{H}$ then degenerates to a vertical line in the torus Π.
2. Through Lemma 6.1 where $f = \psi$ is periodic of period 2π, the set $\mathcal{O}$ is $\mathbf{O}$ where ρ is the ratio ω_2/ω_1 and $\mathbf{O} \cap \mathbf{H}$ is dense in $\{(x, y) \in \mathbf{H} \mid \dot{\psi}(x) \neq \rho\}$ follows. In addition, it is proven above that $\dot{\psi} \neq \rho$ almost everywhere, thus $\overline{\mathbf{O} \cap \mathbf{H}} = \mathbf{H}$, since $\mathbf{H}$ is infinite, thus $\mathbf{O} \cap \mathbf{H}$ is countably infinite. In particular, there is a countably infinite set of $s > 0$ such that $h(s) = 0$.

To complete the proof, we show that Z^+ is countably infinite by proving that $\gamma(Z^+) = \mathbf{O} \cap \mathbf{H} \cap \mathbf{W}^+$ is countably infinite. In a similar manner, it is sufficient to show that $\overline{\mathbf{O} \cap \mathbf{H} \cap \mathbf{W}^+} = \overline{\mathbf{H} \cap \mathbf{W}^+}$ and $\mathbf{H} \cap \mathbf{W}^+$ is an infinite set. If $\beta_2 \neq 0$, denote $\kappa = -\beta_1/\beta_2$, the curve $\mathbf{W}$ then corresponds to the function $y = \tilde{\psi}(x)$ where $\tilde{\psi} = \varphi^{-1}(\kappa\varphi)$ which has the same properties with ψ. The result still holds if $\beta_2 = 0$ since $\mathbf{W}$ degenerates to the vertical line in Π. By Assumption 2.1, $r \neq \kappa$ and it follows that H and W cannot coincide and the determinant of the coefficient matrix of the homogeneous system

$$\begin{aligned} H(x, y) &= 0 \\ W(x, y) &= 0 \end{aligned}$$

is nonzero. Therefore, it has a trivial solution $\varphi(x) = \varphi(y) = 0$, *i.e.* $(\bar{\pi}, \bar{\pi})$ is one intersection between $\mathbf{H}$ and $\mathbf{W}$. Assumption 2.1 is optimal to have Z^+ is infinite. Otherwise, if Assumption 2.1 does not hold, $\mathbf{H} = \mathbf{W}$, thus $\mathbf{H} \cap \mathbf{W}^+ = \varnothing$ and Z^+ is empty.

Assumption 2.1 implies that $\dot{\psi}(\bar{\pi}) \neq \dot{\tilde{\psi}}(\bar{\pi})$ since $\dot{\psi}(\bar{\pi}) = r$ and $\dot{\tilde{\psi}}(\bar{\pi}) = \kappa$, thus the curves are transverse. Moreover, since the signs of the derivatives of ψ and $\tilde{\psi}$ depend on the signs of r and κ, respectively, ψ and $\tilde{\psi}$ are monotonic functions, in which case, $\mathbf{H} \cap \mathbf{W}^+$ is a half of the curve $\mathbf{H}$ which lies in the region $\mathbf{W}^+$ (see Figures 10(a) and 10(b)).

As $\mathbf{W}^+$ is an open set and $\overline{\mathbf{O} \cap \mathbf{H}} = \mathbf{H}$, then $\overline{\mathbf{O} \cap \mathbf{H} \cap \mathbf{W}^+} = \overline{\mathbf{H} \cap \mathbf{W}^+}$. It follows that $\mathbf{O} \cap \mathbf{H} \cap \mathbf{W}^+$ is infinite (Figure 10c.). Hence, Z^+ is infinite. ■

6.2. Internal resonances

The situation is much simpler when the ratio ω_1/ω_2 is rational. All the functions involved are periodic with the same period and the set $\mathbf{O} \cap \mathbf{H}$ is finite or empty. Thus, the set of initial velocity $\{v(s),\ s \in Z\}$ is finite which also means that the set of generalized 1-SPP is finite. Z^+ can be an empty set for instance if $Z = \varnothing$: with the parameters $\alpha_1 = 1$, $\alpha_2 = -1$, and $\omega_1/\omega_2 = 2$, the graph of function $h(s)$ is depicted in Figure 11: 1-SPP do not exist.

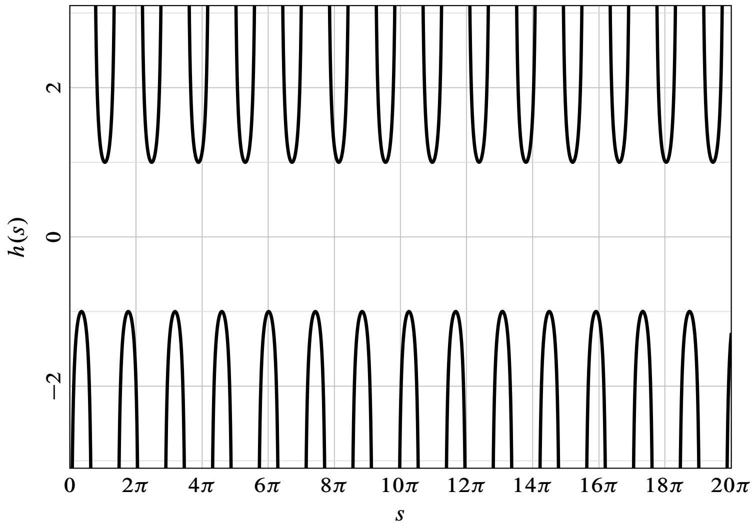

Figure 11: Function $h(s)$ when $\omega_1/\omega_2 = 2 \in \mathbb{Q}$. To be compared to Figure 3.

7. Prestressed structure

In this Section, the structure of the 1-SPP when $d \leq 0$ is addressed. A general argument on the occurrence of the sticking phase is stated in Proposition 7.1. Precisely, sticking phases of unbounded duration arise besides the solutions with finite sticking phases, when the initial velocity of m_1 is zero. 1-SPP for $d < 0$ and $d = 0$ are also explored and illustrated through appropriate numerical examples.

Proposition 7.1 *Assume $d \leq 0$. Then, Theorem 2.1 holds for the sticking phase of finite duration. There is also a unique sticking phase of infinite duration if*

$$u_2(0) = d, \quad \dot{u}_2(0) = 0, \quad u_1(0) = d, \quad \textit{and} \quad \dot{u}_1(0) = 0. \tag{7.1}$$

7.1. Strictly prestressed structure

The system is explored with $d < 0$.

Sticking phase of finite duration From Proposition 7.1, at the beginning of the sticking phase, the initial data are $[\mathbf{u}(0), \dot{\mathbf{u}}(0)]^\top = [d, d, v, 0]^\top$ where $v > 0$. From Equation (5.15) when $d < 0$, it follows that the admissible initial data are found from the set Z^- instead of Z^+. Z^- is also countably infinite as stated in Theorem 2.3. In a manner similar to the case $d > 0$, an infinite set of admissible initial data is expected when $d < 0$.

A 1-SPP is depicted in Figure 12 where $d = -1$; the positive initial velocity is $v \approx 2.26$. With a period $T \approx 5.42$, the sticking phase occurs until $\mathcal{T} \approx 1.58$ then followed by a free flight of duration $s \approx 3.84$.

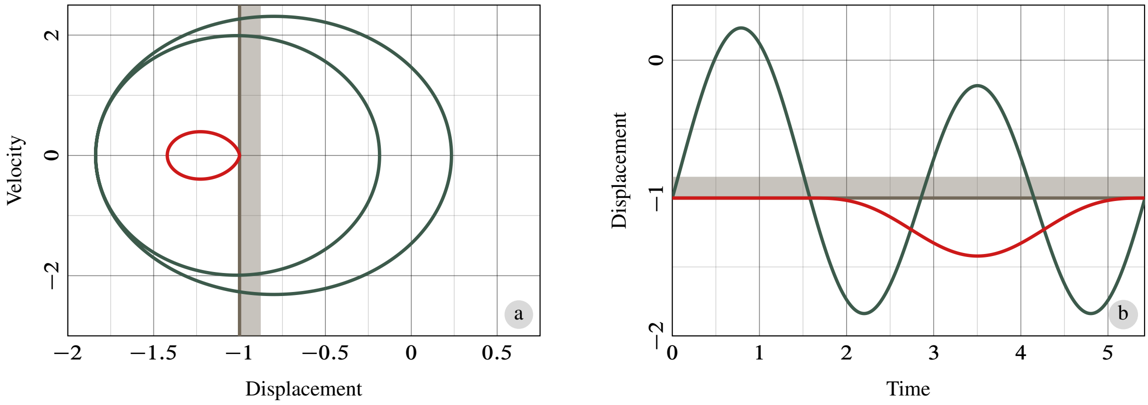

Figure 12: 1-SPP with finite sticking phase for $d < 0$ and $v > 0$. (a) Orbits. (b) Displacements

Sticking phase of infinite duration The corresponding initial data are $[\mathbf{u}(0), \dot{\mathbf{u}}(0)]^\top = [d, d, 0, 0]^\top$. The first mass then follows the oscillation around its new equilibrium at which $u_1(t) = k_2 d/(k_1 + k_2)$. Moreover, 0 is the minimum point of u_1, thus $u_1(t) \geq d$ for all t. By Theorem 2.1, it follows that the sticking phase never ends. This argument is illustrated in Figure 13.

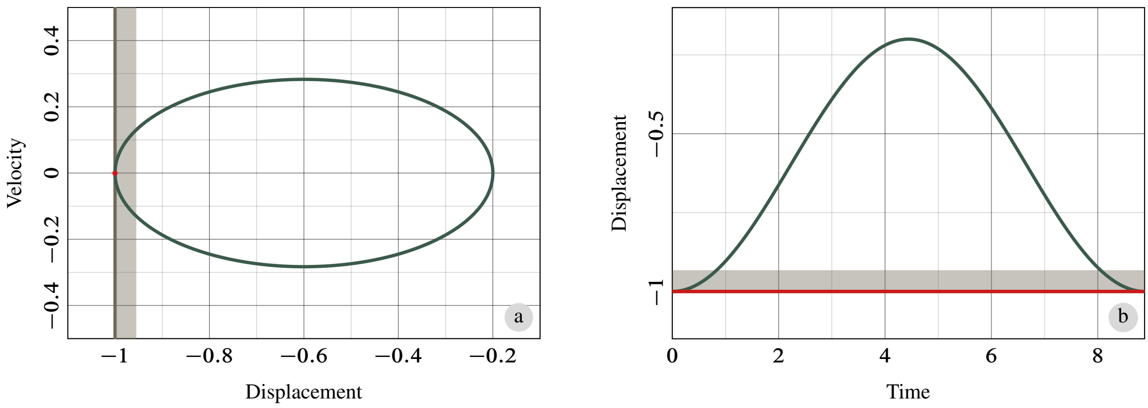

Figure 13: Sticking phase of infinite duration for $d < 0$ and $v = 0$. (a) Orbits. (b) Displacements

7.2. Statically grazing structure

The system is explored with $d = 0$.

Sticking phase of finite duration From Proposition 7.1, the sticking phase with finite duration happens if the initial data satisfy $u_2(0) = 0$, $\dot{u}_2(0) = 0$, $u_1(0) = 0$, and $\dot{u}_1(0) = v > 0$. The set of free flight time s is found from Equation (5.14) where $d = 0$, *i.e.* $v w_1(s) = 0$ and $v w_2(s) = 0$. Hence, v is arbitrarily positive and s satisfies $h(s) = w_1(s) - w_2(s) = 0$ and $w_2(s) = 0$ or

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} \varphi(\omega_1 s/2) \\ \varphi(\omega_2 s/2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By Assumption 2.1, this linear system has the unique solution $\varphi(\omega_1 s/2) = 0$ and $\varphi(\omega_2 s/2) = 0$ where $\varphi(t) = \cot(t/2)$. It follows that

$$\frac{\omega_1}{\omega_2} = \frac{2k+1}{2\ell+1} \quad \text{with} \quad k, \ell \in \mathbb{Z} \tag{7.2}$$

condition which loosely speaking represents half of the rationals. It should be satisfied to observe a sticking phase of finite duration when $d = 0$ while the initial velocity of mass m_1 can be chosen arbitrarily positive. Such a 1-SPP when $\omega_1/\omega_2 = 1/5$ is shown in Figure 14.

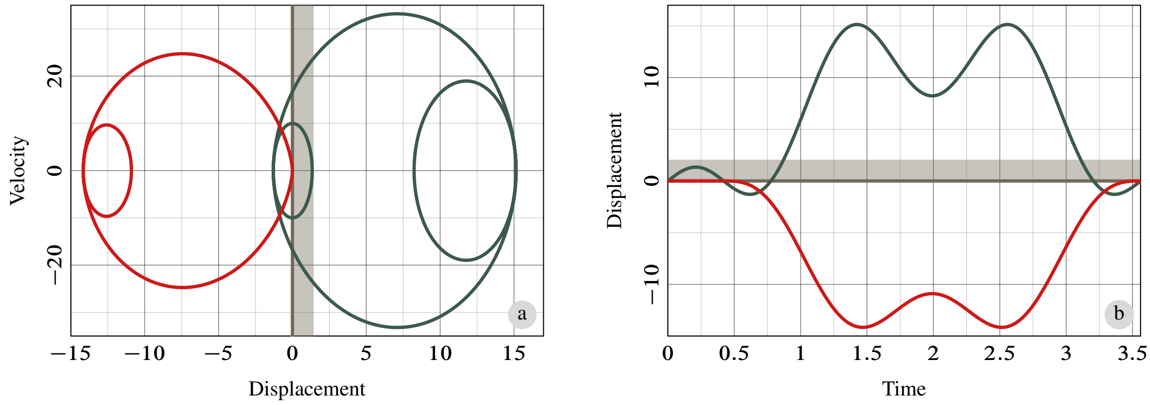

Figure 14: 1-SPP with finite sticking phase for $d = 0$. (a) Orbits. (b) Displacements

Sticking phase with infinite duration The corresponding initial data is $[\mathbf{u}(0), \dot{\mathbf{u}}(0)]^\top = [0, 0, 0, 0]^\top$. The equilibrium $\mathbf{u} \equiv \mathbf{0}$ is a solution in which mass m_2 always grazes with the wall since the two masses stay at their equilibrium points when $d = 0$ and $\dot{u}_1(0) = 0$.

It should be thus noted that generically, there is no 1-SPP except the equilibrium.

8. Conclusion

The free dynamics of a two-degree-of-freedom oscillator subject to a unilateral constraint on one of its masses is investigated. A Newton-like impact law generically emerges in this type of formulation. In this work, periodic orbits with one sticking phase per period (1-SPP) are considered: as such, it is shown that they are independent of the impact law. Moreover, they might not always exist and whenever they do, they are isolated as opposed to one-impact-per-period solutions (1-IPP) known to be organized on manifolds [8]. Also, they cannot be obtained through usual perturbation methods.

The full set of 1-SPP is characterized by only one parameter belonging to a discrete set: the free flight duration. This parameter belongs to a countable set which can be empty, or even infinite in some circumstances. A systematic numerical procedure designed to find all possible 1-SPP is expounded. It involves two numerical steps:

1. finding the roots of an explicit quasi-periodic function, and
2. checking that the corresponding closed-form trajectory satisfies the unilateral condition on the whole period of motion.

Many examples are presented but the mathematical proof of the existence of 1-SPP remains an open problem. The situation is worse: conditions for the non-existence of 1-SPP are provided. However, under generic assumptions on the mass and stiffness matrices of the system, a countable infinite set of initial data including all the initial data of 1-SPP can be exhibited. The closed forms emanating from this set (of initial data) satisfy the unilateral constraint at least near the sticking phase. The prestressed structure is also explored. The picture is similar except that 1-SPP with infinite sticking time are also found.

Extension to n degrees-of-freedom is far from being straightforward, mainly because the symmetry $\mathbf{u}(t) = \mathbf{u}(-t)$, property heavily used in this work, is potentially broken.

9. References

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